A Rubik's Fantasy

Kaylee Huber

Abstract

The Rubik's Cube is a 3-dimensional puzzle that contains 6 sides known as a *face*) and nine *facets* per face. There are 54! possible combinations of rearranging the facets, however, not all rearrangements are types of movements of a Rubik's cube. The total number of possible movements in the Rubik's cube is over 43 quintillion [2]. Due to the cube's nature, many individuals find it challenging to solve. In this paper, we use a 3x3 Rubik's cube to describe and introduce the concept of permutations. Although permutations are a topic of abstract algebra (or an advanced level of mathematics), they are described in a way accessible to everyone regardless of their mathematical background.

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1 Introduction

Permutations can be used in everyday life ranging from creating a four digit password to scheduling classes for a college semester. The methods used to solve a Rubik's cube is one of many applications of permutations. Of all known algorithms to solve the Rubik's cube, we use the Beginner's Method (which is called the cross, first two layers, orientation of the last layer, and the permutation of the last layer method [CFOP]) to illustrate how permutations work. In research, mathematicians have simplified algorithms so that fewer moves are used when solving the cube. Moreover, advancements over time have allowed mathematicians to use computer programming and group theory (a topic of Abstract Algebra) to support their methods.

2 Permutations

In this section, we will introduce permutations, the identity permutation, orbits, and swaps. The concepts introduced in this section can be found in any Abstract Algebra textbook. We refer the reader to [3], [4], [5], [6], and [7].

Definition 1. Given an ordered set A = $(a_1, a_2, \ldots, a_n)$ a **permutation** σ of A is a rearrangement of the elements of A. So $\sigma = (a_{\sigma 1}, a_{\sigma 2}, \ldots, a_{\sigma n})$, where $a_{\sigma 1} = \sigma(a_1)$, $a_{\sigma 2} = \sigma(a_2)$, ..., $a_{\sigma n} = \sigma(a_n)$.

Example 1. Given A = (1, 2, 3, 4), the rearrangement $\sigma = (3, 2, 4, 1)$ is a permutation of A where:

- $\sigma(1) = 3$
- $\sigma(2) = 2$
- $\sigma(3) = 4$
- $\sigma(4) = 1$

σ can be written as

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} \tag{*}$$

which can be also understood as the following:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \sigma(1) & \sigma(2) & \sigma(3) & \sigma(4) \end{pmatrix}$$

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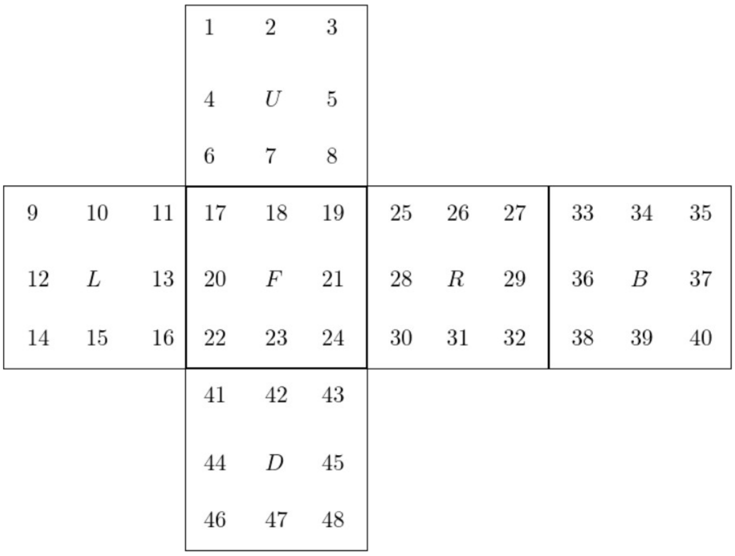

Figure 1: A map of the Rubik's cube, which can be found on [3, Page 72].

Permutations can be used to identify the movements in a Rubik's cube since solving the cube involves rearranging the different facets of the cube [8, Page 3].

The notation in (*) of Example 1 is the most common in abstract algebra texts. The top row of the matrix in this example represents fixed places (or in other words the original positions of the permutation) and the bottom row represents the rearrangement of the top row.

To simplify the matrix notation, we introduce the **symmetric notation** [6, Page 195], which is written as the following:

$$[\sigma(1), \sigma(2), ..., \sigma(n)]$$

Looking back at the permutation from Example 1, using the symmetric notation σ is written as follows:

$$\sigma = [3, 2, 4, 1]$$

When viewing a Rubik's cube, each facet is labeled with a number that defines which position it should be in the cube. In Figure 1, a 3x3 Rubik's cube map can be seen with each facet being labeled with a number (except for the center pieces, which are used to identify which side of the cube you are facing). Although there are a total of 54 facets (when multiplying a side of 9 facets with 6 sides), subtracting the six fixed centerpieces will result in 48 facets as seen in Figure 1. In this figure, the cube is in a fixed position, meaning that the facets are in their proper locations.

2.1 Permutation Types

When learning about permutations, it is important to be familiar with the different strategies that can be used to rearrange the elements of a set. In simpler sets of numbers, it may be easier to find a small pattern as one number may be flipped with another. However, sets containing several numbers may need an advanced strategy to simplify understanding permutation.

2.1.1 Identity Permutation

The **Identity Permutation** (denoted by i) is when the permutation is fixed, or in its original position.

An example of this is as follows:

Example 2. Let A = (1, 2, 3, 4, 5). The identity permutation is $i = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$

In using this to solve a Rubik's cube, the identity permutation is associated when the cube is solved.

2.1.2 Orbits

Definition 2. Let A be a set and k be an element of A. The **orbit** of k under a permutation σ is the sequence $\sigma(k)$, $\sigma(\sigma(k))$, $\sigma(\sigma(\sigma(k)))$...

Remark 1. We denote $\sigma(\sigma(k))$ as $\sigma^2(k)$, $\sigma(\sigma(\sigma(k)))$ as $\sigma^3(k)$, etc...

If the set A is finite, it follows that any orbit is finite and starts and ends at k. In some instances, permutations can contain several orbits. (See Example 3 below.)

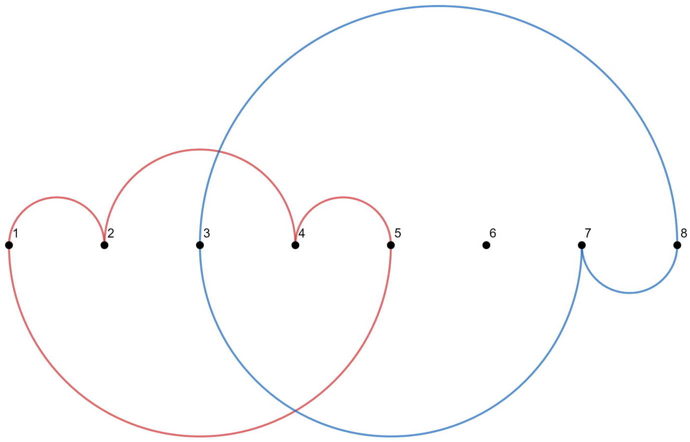

Figure 2: These are the orbits from Example 3, with the red path denoting the first orbit and the blue path denoting the second orbit.

Example 3. Let τ be the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 8 & 5 & 1 & 6 & 3 & 7 \end{pmatrix}$. From this matrix, we can identify three orbits:

1. An orbit that begins and ends with the number 1 since 1 travels through the following sequence:
   - $\tau(1)= 2$
   - $\tau(2)= 4$
   - $\tau(4)= 5$
   - $\tau(5)= 1$
2. An orbit that begins and ends with the number 3 since 3 travels through the following sequence:
   - $\tau(3)= 8$
   - $\tau(8)= 7$

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- $\tau(7) = 3$

3. An orbit that begins and ends with the number 6, but 6 doesn't travel through any sequence of different numbers, since $\tau(6) = 6$.

An orbit consisting of only one element, like the orbit of 6 in Example 3, is called a **trivial orbit**. However, the orbits of 1 and 3 from Example 3 do change numerically, making them **non-trivial orbits**.

Definition 3. A permutation containing only one non-trivial orbit is called a **cycle**.

Notation 1. Let σ be a cycle. Then σ consists of only one non-trivial orbit. Let a denote the smallest element of the orbit and n as the smallest positive integer such that $\sigma^n(a)=$
a. Then the orbit of a consists of a, $\sigma(a)$, $\sigma^2(a)$, $\sigma^3(a)$, ..., $\sigma^{n-1}(a)$. We denote σ by $((a)$,
$\sigma(a)$, $\sigma^2(a)$, ..., $\sigma^{n-1}(a))$.

Remark 2. Cycles are read from left to right.

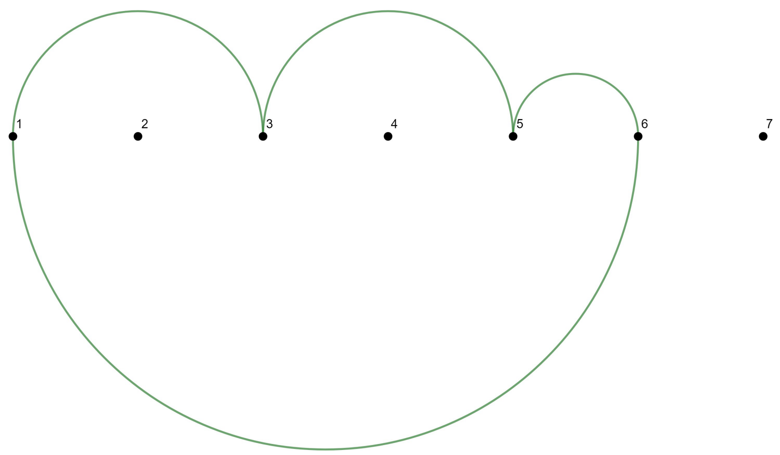

Figure 3: This is the cycle from Example 4

Example 4. Let σ be the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 2 & 5 & 4 & 6 & 1 & 7 \end{pmatrix}$. Using Notation 1 above since σ is a cycle, this permutation becomes $\sigma = (1\ 3\ 5\ 6)$. In other words, this is read as follows:

- 1 goes to 3
- 3 goes to 5
- 5 goes to 6
- 6 goes to 1

Notation 2. $(a_1, a_2, a_3, ..., a_n)$ will be used only to denote cycles.

2.1.3 Permutation Swaps

A **permutation swap** (also known as an adjacent transposition) is when two numbers next to each other are rearranged by switching their positions. (See Example 5.) When applying this permutation, you can continue swapping a number that had been previously swapped. With this permutation, you can only swap the numbers to the left or to the right of the number that you want to swap. After swapping different numbers for some time, you will get a mixed permutation completely different from your original set [9].

Example 5. Using bracket notation, suppose we have the set [1, 2, 3, 4] and we want to swap the number 2 with 3. With the swapping permutation, you will get an outcome of [1, 3, 2, 4]. On the other hand, since 1 and 3 are not next to each other in the original set of [1, 2, 3, 4], we cannot swap them. Now suppose that we wanted to switch the numbers 3 and 1 from the set [1, 3, 2, 4]. Applying the swap of 3 and 1 to [1, 3, 2, 4] will give the permutation [3, 1, 2, 4]. Moreover, if you swap the numbers 1 and 2 in the set [3, 1, 2, 4], you will get [3, 2, 1, 4]. By using swaps, it is possible to rearrange other numbers.

2.2 Composition of Permutations

Definition 4. A **composition** (also called a product) is when two permutations are combined to create a third permutation.

A composition is shown in the following example:

Example 6. Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$. Then,

$$\sigma \cdot \tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

Following the notation of [3], the composition is done from left to right. For example, if $\tau(1) = 2$ and $\sigma(2) = 3$, then $\tau \cdot \sigma(1) = 3$. In other words, 1 goes to 2 under τ, and 2 goes to 3 under σ. Therefore, 1 goes to 3 under $\tau \cdot \sigma$. Completing the remainder of the composition will give you the following product:

$$\tau \cdot \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 2 & 4 \end{pmatrix}.$$

2.2.1 n-Fold Compositions

The **n-fold composition** is the number of times that you multiply a permutation by itself [3, Page 41]. (See Example 7)

Example 7. Assume that we have a permutation σ on a set A. To apply the n-fold composition, multiply σ by itself an n number of times. This can be seen through the following list:

- $\sigma^1 = \sigma$
- $\sigma^2 = \sigma \cdot \sigma$
- $\sigma^3 = \sigma^2 \cdot \sigma$
- $\sigma^n = \sigma^{n-1} \cdot \sigma$

Since the number of permutations is finite, then we have the following theorem that can be found in any Abstract Algebra textbook:

Theorem 1. *Let σ be a permutation of a finite set A. Then there is some integer, $n \geq 0$, which depends on σ, such that $\sigma^n = i$. That is, taking powers of σ, will eventually lead you to the identity permutation.*

This concept is important to the Rubik's cube since the cube has a finite amount of moves. Therefore, doing the same set of moves (starting with a solved cube) a specific number of times will solve the cube.

Definition 5. Let A be a finite set. The **order** of a permutation σ is the smallest positive integer n such that $\sigma^n = i$. This is denoted as ord(σ).

3 Permutation Notation

With permutations, different notations such as the matrix notation and symmetric notation exist and are used for different purposes. In this section, we will introduce the cycle notation.

3.1 Cycle Notation

The **cycle notation** simplifies the symmetric and matrix notations by writing permutations as a product of cycles [3, Theorem 3.3.1].

Example 8. Suppose we have $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$ and we want to multiply this by $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$ If we have $\alpha \cdot \tau$, we would have the following product of cycles:

$$\alpha \cdot \tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = (1, 2)(3, 4)$$

For this cycle product, no numbers are fixed so (1, 2, 3, 4) are included in the notation. However, fixed numbers can be removed when writing a permutation as a product of cycles, which can be seen in the following example:

3.1.1 Example

Let $\omega = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$ and let $\mu = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$. Then, $\omega \cdot \mu = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$.

In symmetric form, we would typically write this in notation form as [1, 3, 4, 2]. However, we can see that 1 remains fixed under $\omega \cdot \mu$. Therefore, we can write the product as (2, 3, 4).
In application to the mathematics of a Rubik's cube, these permutation notations become important as they make the permutation products easier to read.

3.1.2 Transpositions

Similar to swapping permutations, **transpositions** take two numbers and switch them without the restriction of the numbers being next to each other, as opposed to the swap permutation [9].

Example 9. Let $\gamma = [2, 5, 8, 6, 1, 4, 3, 7]$ and suppose we want to switch the positions of 2 and 3. Using a transposition cycle, we get $\mu = [3, 5, 8, 6, 1, 4, 2, 7]$.

Theorem 2. *Every transposition can be obtained from a product of swaps.*

Proof. Let A = $[a_1, a_2, ..., a_n]$ be an ordered set and $a_i, a_j \epsilon A$, $i < j$. We will show that $\sigma(A) = [a_1, ..., a_j, ..., a_i, ..., a_n]$ is a product of swaps.

Let A = $[a_1, \ldots, a_i, a_{i+1}, a_{i+2}, \ldots, a_{j-1}, a_j, \ldots, a_k]$. Swapping a_i with a_{i+1}, we get

$$\sigma_1(A) = [a_1, \ldots, a_{i+1}, a_i, a_{i+2}, \ldots, a_{j-1}, a_j, \ldots, a_n].$$

Observe that a_i and a_{i+2} are next to each other.

Swapping a_i with a_{i+2}, we get

$$\sigma_2(\sigma_1(A)) = [a_1, \ldots, a_{i+1}, a_{i+2}, a_i, \ldots, a_{j-1}, a_j, \ldots, a_n].$$

Similar to the last swap, we notice that a_i and a_{i+3} are now next to each other, meaning that we can repeat the same operation: swapping a_i with a_{i+3}. Repeating this process a $j - i$ amount of times, we get

$$\sigma_{j-i}(\sigma_{j-i-1}(...(\sigma_1(A))...)) = [a_1, \ldots, a_{i+1}, a_{i+2}, \ldots, a_{j-1}, a_j, a_i, a_{j+1}, \ldots, a_n].$$

So, after completing a $j - i$ amount of swaps, a_i and a_j are next to one another.

Next, we swap a_j in the reverse order that we did with a_i. Swapping a_j a $j - (i + 1)$ amount times will move a_j to be where a_i once was. Then σ is the product of all these swaps. That is,

$$\sigma(A) = [a_1, \ldots, a_j, a_{i+1}, a_{i+2}, ..., a_{j-1}, a_i, a_{j+1}, ...a_n]$$

□

3.1.3 Disjoint Cycles

Two permutations are **disjoint** if they permute two or more sets with no common elements.

Example 10. Let σ=(2, 5, 3) and τ= (1, 8, 9) be permutations. We can see that they are disjoint because they have no numbers in common.

However, if we let μ= (1, 3, 5) and ν= (3, 2, 6), we can conclude that these permutations are not disjoint because they both share the number 3.

With the inclusion of disjoint cycles, the following theorem can be used (the proof can be found on [3, Page 49, Theorem 3.3.1]):

Theorem 3. *Every permutation is the product of disjoint cycle permutations.*

From Theorem 3, every movement of a Rubik's cube can be written as a product of disjoint cycles, which will simplify the permutations of the cube as sections of the cube become fixed.

4 Rubik's Cube Notation

Now that we know the elementary components of permutations, it is important to learn its application to the different elements of a 3x3 Rubik's cube. This section will discuss the cube's faces, pieces [10, Pages 4-5], and number assignment.

4.1 Faces

The faces (or each side of the Rubik's cube) are assigned a specific color and they can either be red, orange, green, blue, yellow, or white. Each color has opposite sides, which include white and yellow, green and blue, and red and orange. When holding the cube, each face has a specific side and can be identified with any color (See List 1.)

With these six faces, each one is assigned a position in the cube to define what specific face needs to be moved. In the instance that is used throughout this paper, the upper face is the white face of the cube, the left face is the orange face, the front face is the green face, the right face is the red face, the back face is the red face, and the down face is the yellow face.

Remark 3. Note that the faces of the cube are identical for any 3x3 Rubik's cube and can be found on any Rubik's cube website, journal and/or book.

4.2 Rotation

As a cube's face is rotated, its type of rotation can be denoted with the faces notation that can be found in List 1. In general, any normal abbreviation (such as U) denotes that you will move the face of the cube labeled U by 90 degrees in a clockwise motion. Moreover, some algorithms will contain moves with a number in front of them (e.g. 2L). When this occurs, you turn the corresponding face of the cube 180 degrees, either in a counterclockwise or clockwise motion.

However, algorithms of the Rubik's cube do contain *inverse* moves and are usually notated with an apostrophe (e.g. L') When an inverse is listed in an algorithm, the assigned face of the cube is moved 90 degrees in a counterclockwise motion.

Remark 4. As rearranging the facets of an unsolved face may be challenging, it is important to ensure that face F is the one directly facing you. Furthermore, you are allowed to rotate the cube to view the other faces while solving it, which does not change the labeling of the cube.

- Up (U)
- Left (L)
- Front (F)
- Right (R)
- Back (B)
- Down (D)

List 1: Faces Notation

4.3 Pieces of the Cube

When looking at the mechanics of the cube, it is important to understand the significance of the different pieces: the centers, edges, and corner pieces [11].

4.3.1 Centers

The Rubik's cube contains 6 sides, 6 colors, and 6 centers. The centers can never be moved and serve the purpose of defining what color that specific face needs to be.

4.3.2 Edges

Unlike centers, the edge pieces contain two different colored sides. The purpose of the edge pieces is to connect the centers. When connected and aligned to the colors of the different centers, it will create a cross pattern, which will be described later in this paper.

4.3.3 Corners

While edges connect to two sides and centers of the Rubik's cube, the corner pieces contain three sides with three different colors. Not only do the corner pieces have to connect to the edge and centers but they connect three faces of the cube. The corner pieces are important in solving the cube as they help solve one or more faces of the cube at a time.

4.4 Numbering the Cube

When numbering the cube, we count 48 facets instead of 54 facets, since the center facets of each face are stationary. Moving these centers is equivalent to making two movements that are either up and down, left or right, or front and back. To number the facets properly, it is important to number them from left to right starting on the first row of the upper face. When the first row is marked, you can start on the second row of the same face, numbering from left to right. This process is also used on the third row of the cube. As soon as the upper face contains a number assignment for its facet, you will continue to do the same process for the other faces in the face order that was listed in List 1. When finished, your cube should be numbered as shown in Figure 1. Moreover, the numbering of the cube can be written out as a permutation, since the elements in

the permutations will be used to label the elements of the cube. (For instance, the first element in the permutation will be placed where 1 is in Figure 1, the second element will be placed where 2 is in Figure 1, etc.)

When rotating each face 90 degrees clockwise on a fixed cube, the following permutations are produced (See [3, Page 72]).

- U= (1, 3, 8, 6) (2, 5, 7, 4) (9, 33, 25, 17) (10, 34, 26, 18) (11, 35, 27, 19)
- L= (9, 11, 16, 14) (10, 13, 15, 12) (1, 17, 41, 40)(4, 20, 44, 37) (6, 22, 46, 35)
- F= (17, 19, 24, 22) (18, 21, 23, 20) (6, 25, 43, 16) (7, 28, 42, 13) (8, 30, 41, 11)
- R= (25, 27, 32, 30) (26, 29, 31, 28) (3, 38, 43, 19) (5, 36, 45, 21) (8, 33, 48, 24)
- B= (33, 35, 40, 38) (34, 37, 39, 36)(3, 9, 46, 32) (2, 12, 47, 29) (1, 14, 48, 27)

- D= (41, 43, 48, 46) (42, 45, 47, 44) (14, 22, 30, 38) (15, 23, 31, 39) (16, 24, 32, 40)

Remark 5. The matrix representation of these permutations is given in Figure 4. The cycles are color-coded where the same colored numbers permute among themselves. How- ever, the light blue colored numbers do not permute among themselves since they are fixed.

In addition to this, we include the inverses of each of the moves from this list, which is as follows:

- U'= (6, 8, 3, 1) (4, 7, 5, 2) (17, 25, 33, 9) (18, 26, 34, 10) (19, 27, 35, 11)
- L'= (14, 16, 11, 9) (12, 15, 13, 10) (40, 41, 17, 1) (37, 44, 20, 4) (35, 46, 22, 6)
- F'= (22, 24, 19, 17) (20, 23, 21, 18) (16, 43, 25, 6) (13, 42, 28, 7) (11, 41, 30, 8)
- R'= (30, 32, 27, 25) (28, 31, 29, 26) (19, 43, 38, 3) (21, 45, 36, 5) (24, 48, 33, 8)
- B'= (38, 40, 35, 33) (36, 39, 37, 34) (32, 46, 9, 3) (29, 47, 12, 2) (27, 48, 14, 1)
- D'= (46, 48, 43, 41) (44, 47, 45, 42) (38, 30, 22, 14) (39, 31, 23, 15) (40, 32, 24, 16)

Throughout this paper, we will use these lists to solve the cube numerically by fixing each number into its correct position.

5 Applying Permutations to the Rubik's Cube

Now that we understand the different components and purposes for each piece of the cube, we can start applying permutations to the Rubik's cube to identify how specific algorithms fix specific pieces of the cube. For this instance, we will be using the beginner's method of solving the cube (also known as CFOP)[12].

$$U=\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 & 27 & 28 & 29 & 30 & 31 & 32 & 33 & 34 & 35 & 36 & 37 & 38 & 39 & 40 & 41 & 42 & 43 & 44 & 45 & 46 & 47 & 48 \\ 3 & 5 & 8 & 2 & 7 & 1 & 4 & 6 & 33 & 34 & 35 & 12 & 13 & 14 & 15 & 16 & 9 & 10 & 11 & 20 & 21 & 22 & 23 & 24 & 17 & 18 & 19 & 28 & 29 & 30 & 31 & 32 & 25 & 26 & 27 & 36 & 37 & 38 & 39 & 40 & 41 & 42 & 43 & 44 & 45 & 46 & 47 & 48 \end{pmatrix}$$

$$L=\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 & 27 & 28 & 29 & 30 & 31 & 32 & 33 & 34 & 35 & 36 & 37 & 38 & 39 & 40 & 41 & 42 & 43 & 44 & 45 & 46 & 47 & 48 \\ 17 & 2 & 3 & 20 & 5 & 22 & 7 & 8 & 11 & 13 & 16 & 10 & 15 & 9 & 12 & 14 & 41 & 18 & 19 & 44 & 21 & 46 & 23 & 24 & 25 & 26 & 27 & 28 & 29 & 30 & 31 & 32 & 33 & 34 & 6 & 36 & 4 & 38 & 39 & 1 & 40 & 42 & 43 & 37 & 45 & 35 & 47 & 48 \end{pmatrix}$$

$$F=\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 & 27 & 28 & 29 & 30 & 31 & 32 & 33 & 34 & 35 & 36 & 37 & 38 & 39 & 40 & 41 & 42 & 43 & 44 & 45 & 46 & 47 & 48 \\ 1 & 2 & 3 & 4 & 5 & 25 & 28 & 30 & 9 & 10 & 8 & 12 & 7 & 14 & 15 & 6 & 19 & 21 & 24 & 18 & 23 & 17 & 20 & 22 & 43 & 26 & 27 & 42 & 29 & 41 & 31 & 32 & 33 & 34 & 35 & 36 & 37 & 38 & 39 & 40 & 11 & 13 & 16 & 44 & 45 & 46 & 47 & 48 \end{pmatrix}$$

$$R=\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 & 27 & 28 & 29 & 30 & 31 & 32 & 33 & 34 & 35 & 36 & 37 & 38 & 39 & 40 & 41 & 42 & 43 & 44 & 45 & 46 & 47 & 48 \\ 1 & 2 & 38 & 4 & 36 & 6 & 7 & 33 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 3 & 20 & 5 & 22 & 23 & 8 & 27 & 29 & 32 & 26 & 31 & 25 & 28 & 30 & 48 & 34 & 35 & 45 & 37 & 43 & 39 & 40 & 41 & 42 & 19 & 44 & 21 & 46 & 47 & 24 \end{pmatrix}$$

$$B=\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 & 27 & 28 & 29 & 30 & 31 & 32 & 33 & 34 & 35 & 36 & 37 & 38 & 39 & 40 & 41 & 42 & 43 & 44 & 45 & 46 & 47 & 48 \\ 14 & 12 & 9 & 4 & 5 & 6 & 7 & 8 & 46 & 10 & 11 & 47 & 13 & 48 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 & 1 & 28 & 2 & 30 & 31 & 3 & 35 & 37 & 40 & 34 & 39 & 33 & 36 & 38 & 41 & 42 & 43 & 44 & 45 & 32 & 29 & 27 \end{pmatrix}$$

$$D=\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 & 27 & 28 & 29 & 30 & 31 & 32 & 33 & 34 & 35 & 36 & 37 & 38 & 39 & 40 & 41 & 42 & 43 & 44 & 45 & 46 & 47 & 48 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 22 & 23 & 24 & 17 & 18 & 19 & 20 & 21 & 30 & 31 & 32 & 25 & 26 & 27 & 28 & 29 & 38 & 39 & 40 & 33 & 34 & 35 & 36 & 37 & 14 & 15 & 16 & 43 & 45 & 48 & 42 & 47 & 41 & 44 & 46 \end{pmatrix}$$

Figure 4: Permutations

5.1 Scrambling the Cube

Since we are trying to solve the Rubik's cube, we will create an unsolved cube with random face movements. For this instance, we used a combination of moves to undo the cube's fixed position, which is as listed:

$$\begin{gathered} F2 * L' * D' * R2 * F' * U2 * D2 * L * U' * F * U2 * B * R2 * D2 * R2* \\ U2 * F * U2 * B2 * L2 * D2 \end{gathered} \tag{1}$$

With this, we can identify the product of cycles for the first permutation applied to the cube, which can be found below:

$$\begin{gathered} (1, 11, 16, 48, 40, 25, 27, 9, 6, 22, 32, 14, 19, 3, 35, 17, 41, 38, 46, 8, 33)(2, 45) \\ (4, 47, 36, 10, 39, 29)(5, 37, 42, 26, 12, 23)(7, 44, 20, 21)(13, 28, 18, 15) \\ (24, 43, 30)(31, 34) \end{gathered} \tag{2}$$

As we continue applying different algorithms to this cycle product, we will find that it will change in terms of its length and number order since the algorithms will fix certain facets into place and rearrange the order of the other numbers.

5.2 Solving the First Layer

To solve the first layer of the Rubik's cube, we have to use a set of algorithms that fix the edge and corner pieces on the upper layer, which involves the white face (U), orange face (L), green face (F), red face (R), and blue face (B). For this to happen, we have to break up the section of solving the first layer into two parts.

5.2.1 The White Cross

In the first part of solving the top layer, we fix the white edge pieces to create what is known as the

White Cross Permutation. To do this mathematically, we multiply the permutation labeled (1) by the corresponding permutations from Page 10 of this paper. The different movements that we used to create the white cross from the scramble are as follows (which is read from left to right)

$$D * 2F * 2B * R' * 2L$$

From this, we get the following product of cycles:

(1, 32, 43, 22, 8, 17, 46)(6, 14, 35, 48, 30, 16, 19)(9, 38, 24, 41, 25, 11, 40)
(12, 29, 44, 45, 47, 20, 37, 36, 15, 31, 39, 13)(21, 42, 28, 23)

Now we can identify that the following facets that are now fixed include 2, 3, 4, 5, 7, 10, 18, 26, 27, 33, and 34 because they are not present in the product of cycles above this paragraph, and this concept can be applied to future cycles throughout this paper.

Remark 6. 3, 27, and 33 do not make the white cross because it is part of a corner piece, yet it becomes fixed when the following movements are applied to make the white cross.

5.2.2 First Layer Corners

Now that a white cross pattern is identified numerically, the rest of the upper face of the cube can be solved. To do this, we apply a down movement each time before applying the Sledgehammer Algorithm (which is a part of the Beginner's Method, for more information see [13]), which can be seen below:

- D* L'* F* L* F
- D* F'* R* F* R'
- D'* B'* L * B * L'

After completing these movements, you will get the following product of cycles:

(12, 29, 21, 45, 20, 44) (13, 15, 37, 36, 28, 31) (14, 46, 40) (16, 38, 41, 32, 22, 48) (23,42) (39, 47)

By using these movements and algorithms, we can see that the upper face is now fixed and the first horizontal rows of the left, front, right, and back faces are fixed. Now, we will be able to solve the second layer of the cube.

5.3 Solving the Second Layer

Because the centers of the faces are stationary, we will only need to solve the edges of the second rows. We can use the First Two Layer Algorithm (also known as F2L [14]) to solve this layer, which is as follows:

- D'* F'* D'* F* D* L* D* L'
- F* D* F'* D'* R'* D'* R
- D* B'* D'* B * D * R * D * R'
- D2 * L'* D'* L* D* B* D* B'

After applying the F2L algorithms to its current state, we get the following cycle product:

$$(15, 42, 45, 39)(16, 43, 41, 30, 22, 24)(23, 31, 42, 45)(38, 41, 40)(39, 47)$$

With the second layer being solved, it is important to know that all the edge pieces in the center layer are fixed. After completing this layer, we can begin solving the third layer of the cube.

5.4 Solving the Third Layer

To solve this layer, we will break up the methods of solving the third layer into three parts.

5.4.1 The Yellow Cross

Similar to the first layer, the Beginner's Method uses an algorithm that helps create a cross shape on the down face of the cube. To create this cross, we use the following algorithm:

$$F * D * L * D' * L' * F'$$

The algorithm identified above is known as the Orientation of the Last Layer (OLL). There are 57 existing OLL algorithms, with the one listed above being the 44th algorithm [15]. After applying this algorithm twice to the current state of the cube, we create the yellow cross, and its cycle product can be written as follows:

$$(14, 46, 40)(16, 30)(22, 43)(23, 31)(24, 41)(32, 48, 38)(42, 45)$$

As the cross is created on the yellow/down face, the facets of the yellow edge pieces are not fixed, since edge pieces have to be fixed with both the yellow center and the centers found in the middle layer. The method for solving this issue will be described in the third part of solving the third layer.

5.4.2 Solving the Yellow Face

Since we already have the yellow cross, we need to solve the corners of the yellow side of the cube. We can get these corners to line up with the **Sune Algorithm** [16]. As this algorithm is used, however, we incorporate extra moves so we can apply the algorithm on either the front, back, left, or right side of the cube.

For this instance, we fixed the front face of this cube so we could apply this algorithm. The Sune algorithm is as follows

$$L * D * L' * D * L * D2 * L'$$

The combinations of Sune that we apply to the cube are listed in the following order:

- L * D* L'* D* L* D2* L'
  (S)
- D'* L* D* L'* D* L* D2* L'
  (D'S)
- D2* L* D* L'* D* L* D2* L'
  (D2S)

After applying the Sune, D' Sune, and D2 Sune algorithms, the cycle product of the cube is as follows:

$$(14, 30, 38, 22)(16, 40, 24, 32)(23, 39)(41, 46, 43, 48)(42, 47)$$

Similar to the yellow cross, the facets of the corners of the third row are not fixed, meaning that more algorithms must be applied to the cube to fix them into their positions.

5.4.3 Solving the Remaining Facets

As the yellow face is fixed, we must solve the remaining corner and edge pieces of the third row of the left, front, right, and back faces. To do this, we have to use the first version of the Aa permutation in the Permutation of the Last Layer Algorithm (PLL) [17]. The Aa permutation is listed below:

$$L' * F * L' * B2 * L * F' * L' * B2 * L2$$

To apply this algorithm to the cube, we altered the algorithm so we could solve it on the same fixed front face. The alterations of the algorithms are listed below:

- D2* L'* F* L'* B2* L* F'* L'* B2* L2* D'
- F2* D'* L'* R* F2* L* R'* D'* F2
- D'* F2* D'* L'* R* F2* L* R'* D'* F2* D

When applying these algorithms in the specific order that they are listed to the cube, we will end up with no cycle product because all positions of the cube are fixed. Therefore, the cube is solved.

6 Least Common Multiples

Definition 6. A **Least common multiple** (or LCM) is the smallest multiple number that two or more numbers have in common with one another (e.g. the LCM of the numbers 2 and 3 is 6).

By using the definition or Least Common Multiples, the products of cycles, and the orders of the product of cycles, the following question can be answered:

Question 1. Using Theorem 1, let σ be an unsolved cube that was created using n movements. What is the minimum k such that $\sigma^k = i$, where i denotes the identity permutation?

In other words, this question asks how many times we can use a specific algorithm repeatedly to solve the cube. Moreover, the LCM comes into play with the cycle products as it determines the LCM of the orders of the cycles. The following theorem answers this question:

Theorem 4. *The order of a permutation of a finite set written in disjoint cycle form is the least common multiple of the lengths of the cycles [7, Page 106].*

The next example applies the definition of LCM and Theorem 4.

Example 11. Assume we take a fixed Rubik's Cube and apply the moves U and L to it. Doing this will result in the following product of cycles:

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U·L = (1, 3, 8, 22, 46, 35, 27, 19, 16, 14, 9, 33, 25, 41, 40)·(2, 5, 7, 20, 44, 37, 4)·(6, 17, 11)·(10, 34, 26, 18, 13, 15, 12)
If we denote this product as U·L = $\sigma_1 \cdot \sigma_2 \cdot \sigma_3 \cdot \sigma_4$, then

- ord(σ_1) = 15
- ord(σ_2) = 7
- ord(σ_3) = 3
- ord(σ_4) = 7

From these numbers, we get an LCM of 105, meaning that you must repeat the moves U and L 105 times to fix the cube.

The computed cycle product of (1) from Section 5.1 is as seen below:

(1, 11, 16, 48, 40, 25, 27, 9, 6, 22, 32, 14, 19, 3, 35, 17, 41, 38, 46, 8, 33) (2, 45) (4, 47, 36, 10, 39, 29) (5, 37, 42, 26, 12, 23) (7, 44, 20, 21) (13, 28, 18, 15) (24, 43, 30) (31, 34)

The order of each cycle is also found in the following list:

- (1, 11, 16, 48, 40, 25, 27, 9, 6, 22, 32, 14, 19, 3, 35, 17, 41, 38, 46, 8, 33) has **21 numbers**, so the order is **21**
- (2, 45) has **2 numbers**, so the order is **2**
- (4, 47, 36, 10, 39, 29) has **6 numbers**, so the order is **6**
- (5, 37, 42, 26, 12, 23) has **6 numbers**, so the order is **6**
- (7, 44, 20, 21) has **4 numbers**, so the order is **4**
- (13, 28, 18, 15) has **4 numbers**, so the order is **4**
- (24, 43, 30) has **3 numbers**, so the order is **3**
- (31, 34) has **2 numbers**, so the order is **2**

From this list, we can see that the following numbers that we have to find the LCM of are 2, 3, 4, 6, and 21. The LCM of these numbers is calculated to be 84. Moreover, you must repeat the algorithm 84 times before fixing the cube's facets

6.1 Further Information about LCM and other Algorithms

Similar to the previous example, you can find the LCM of any other Rubik's cube algorithm or set of movements using the following steps:

1. Find the permutation set of the set of movements/algorithms that you are using by using two-row matrices.
2. Simplify your two-row matrices into a product of cycles.
3. Find orders of each cycle product group.

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4. Find the LCM of the orders.

After finding the LCM of this algorithm/set of movements, you can then conclude that you need to use the LCM times that specific algorithm or set of movements to fix the facets of the cube.

7 Conclusion

The application of mathematics in everyday life can allow individuals to better comprehend mathematical concepts. After reading this paper, readers can understand what permutations are as it is introduced with the use of a 3x3 Rubik's cube. In addition, the paper and glossary's descriptions of mathematical terms can be described to anyone regardless of their mathematical background. Finally, learning each purpose of the Rubik's cube's pieces will allow users to better understand the mathematical processes of solving the Rubik's cube. While this paper doesn't teach the reader how to solve the Rubik's cube, it allows readers to learn how math can be used to study the different possible combinations available for solving the cube.

Glossary

Beginner's Method A method consisting of basic algorithms used to solve a Rubik's Cube. 1

Compositions The combinations of two permutations to create a third permutation; the product of two permutations. 5

Cycle A permutation that only contains one non-trivial orbit. 4

Cycle Notation The simplification of the symmetric and matrix notations by rewriting permutations as a product of cycles. 6

CFOP An acronym of the algorithms used in the Beginner's Method, with each of the algorithms being the cross, first two-layer method, orientation of the last layer, and permutation of the last layer. 1

Disjoint Two or more sets with no common elements. 7

Elements The numbers that exist in a permutation. 1

Face One of the six sides of a 3x3 Rubik's Cube. 1

Facets Singular-dimensioned squares that create a cube's face. 1

First Two Layer Algorithm The algorithm used to fix the edges of the second row of the Rubik's cube. 13

Fixed Another term to describe the state of the elements of the identity permutation. 2

Identity Permutation When the permutation's elements are in their original positions. 3

Least Common Multiple The smallest multiple number that two or more numbers have in common with one another. 15

Non-Trivial Orbit An orbit that contains more than one element. 4

Orbit The path that a single element goes through in the permutation. 3

Order The smallest positive integer n of the permutation σ such that $\sigma^n = i$. 6

Orientation of the Last Layer The algorithm used to create the yellow cross on the down face of the Rubik's cube. 13

Permutation A permutation of a set A is a rearrangement of the elements of A. 1

Permutation of the Last Layer The algorithm used to fix the remaining facets of the third layer of the Rubik's cube. 14

Permutation Swap When two numbers next to one another are rearranged by switching their positions. 5

Rubik's Cube A three-dimensional puzzle that contains six faces and nine facets per face. 1

Sledgehammer Algorithm The algorithm applied to the Rubik's cube to fix the corners of the first layer. 12

Sune Algorithm The algorithm used to fix the corners of the cube's down face. 14

Symmetric Notation The notation used to simplify common matrices notation. 2

Transpositions A permutation where two numbers are switched without them being next to one another. 7

Trivial Orbit An orbit that consists of only one element. 4

White Cross Permutation A permutation that fixes the white edges of a Rubik's cube. 12

Yellow Cross The outcome used by the Orientation of the Last Layer algorithm to fix the facets of the down face of the cube. 13

n-Fold Composition The number of times that a permutation is multiplied by itself. 5

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